International Journal of Theoretical Physics, Vol. 44, No. 7, July 2005 (© 2005)
DOI: 10.1007/s10773-005-7077-4

Master-Equations for the Study of Decoherence

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Received August 14, 2004; accepted October 4, 2004

Different structures of master-equation used for the description of decoherence of a microsystem interacting through collisions with a surrounding environment are considered and compared. These results are connected to the general expression of the generator of a quantum dynamical semigroup in presence of translation invariance recently found by Holevo.

KEY WORDS: decoherence; master-equations; translational invariance; quantum theory; quantum-dynamical semigroup; Lèvy–Khintchine formula; open system.

1. INTRODUCTION

In recent times the word decoherence has become quite fashionable in order to describe a range of utterly different physical situations, which however all exhibit a common qualitative feature: a quantum system, due to its unavoidably imperfect isolation from the surrounding environment, shows in its dynamical evolution the suppression of typical quantum coherence properties, such as interference capability. Although the subject is in rapid evolution, a nice recent presentation of the field, anchored to the robust background of the theoretical description of open quantum systems, can be found in (Breuer and Petruccione, 2002).

The basic ideas are actually very old and as it was recently stressed in (Dubé and Stamp, 2001) can be essentially traced back to the first studies on the measurement problem in quantum mechanics in the 50's. These studies in which the main concepts related to decoherence already appeared has led by now to relevant improvements in the formulation of quantum mechanics, going beyond Dirac's presentation and leading to the new concept of effect, positive operator valued measure, operation and instrument, also disvealing most fruitful and interesting connections with the theory of stochastic processes; it appears indeed that many useful clues in studying the theory of quantum systems can be obtained from classical probability theory, rather than from the usual correspondence with classical mechanics. These more advanced and flexible tools in the description of quantum

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systems and their dynamics are by now extensively used in quantum information and communication theory.

Indeed the common root between decoherence and the measurement problem, that is the interaction between a microsystem and a macrosystem, obviously indicates that concepts, techniques and tools originating in the realm of foundations of quantum mechanics will prove an essential ingredient in the actual study of decoherence. In this respect a relevant distinction is to be pointed out. One thing is the loss of quantum coherence for a microsystem interacting with some macrosystem, another thing the classical behavior which macrosystems actually exhibit, thus allowing for an objective description. Though the latter phenomenon can be thought of as a consequence of the first, enhanced by the huge number of degrees of freedom pertaining to a macrosystem, the two physical situations are actually very different and it might well be the case that different approaches should be devised, the connection being not necessarily trivial as is often conjectured or implicitly assumed. In particular the description of macrosystems should rely on a suitable development of quantum statistical mechanics, which extended to non-equilibrium situations could allow for the appearance of a classical behavior for a subset of observables, possibly giving useful insights in the description of decoherence for a microsystem (Lanz *et al.*, 2000, 2002). Since the connection between the phenomenon of decoherence and the measurement problem has been touched upon, it is important to stress that decoherence is not a solution to the aforementioned problem. This incorrect viewpoint is often implicitly or explicitly assumed, however as recently most clearly shown in the standard framework of Dirac's formulation of quantum mechanics (Bassi and Ghirardi, 2000), even supposing that due to interaction with the environment the combined system composed by microsystem and apparatus would end up in a statistical mixture with respect to some pointer basis, there is no reason for this basis to be factorisable and typically in the case of a measurement apparatus the combined system will not exhibit macroscopically distinct states.

The reason for the recent renewal of interest in decoherence is twofold: first enormous experimental progress has been made in dealing in a controlled way with microsystem, also engineering superposition states which might be particularly sensitive to decoherence effects, and second decoherence is perhaps the worst enemy when it comes to the physical implementation of quantum computers. These two major motivations push current research work in the two related directions of both quantitatively understanding and avoiding decoherence. Up to now most theoretical models of decoherence have been chosen rather because of their solubility, than because of their adherence to realistic physical models. The very universality which is often expected and advocated for the phenomenon actually relies on suitable modeling for reservoir and interaction, so that specific properties of system, bath and interaction should be of relevance in explaining the sensitivity to the different environmental couplings which actually appears in experiments, as recently stressed for example in (Dubé and Stamp, 2001; Anastopoulos and Hu, 2000; Anastopoulos, 2002). Real progress in modeling and understanding of the phenomenon depends on a detailed description of the physical system and of its dynamics. In this spirit in the following we will try to focus on structures of master-equation (ME) which apply to the description of decoherence of a neutral massive microsystem coupled to the environment due to collisions with environmental particles. Many proposals have been put forward in the literature and we will briefly outline the relationships among the different models. The Markovian description level pertaining to these ME is certainly not the most general physical picture, but seems appropriate to this kind of dynamics. It is of course of particular interest to investigate how and under which conditions ME do emerge from a more refined description of the reduced dynamics of the microsystem, as has recently been done in (Ankerhold, 2003) with reference to the path integral approach, where non-Markovian effects and strong coupling can be taken into account, but the noise is essentially bound to be Gaussian.

Useful insights in the structure of the Markovian ME can be gained from the mathematical characterization of generators of quantum dynamical semigroups (Alicki and Fannes, 2001; Alicki, 2002a) recently given by Holevo under the further requirement of translation invariance (TI) (Holevo, 1995, 1996). This characterization arises from a deep analogy with the classical Lévy–Khintchine formula for the characteristic function of a Lévy process, i.e., a stochastic process homogeneous in space and time, thus having independent and stationary increments; it provides a more detailed description for the possible structure of the generator of the dynamics than the Lindblad result.

The paper is organized as follows: in Section 2. we deal with different ME for the description of decoherence induced by collisions; in Section 3. we compare these results with the structures arising in presence of TI.

2. MODELS OF DECOHERENCE INDUCED BY COLLISIONS

A useful model for the description of decoherence was first obtained in (Joos and Zeh, 1985)

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathrm{JZ}\,[\hat{\rho}] \qquad \mathrm{JZ}\,[\hat{\rho}] = -\Lambda \sum_{i=1}^{3} [\hat{\mathbf{x}}_i, [\hat{\mathbf{x}}_i, \hat{\rho}]], \tag{1}$$

where $\hat{\rho}$ is the statistical operator associated to the microsystem and $\hat{H}$ its Hamiltonian which here and in the sequel we take to be that of a free particle. This ME describes the dynamics of the center of mass of the microsystem due to scattering with an incoming particle flux. It allows in a straightforward way for the introduction of a typical decoherence time after which off-diagonal matrix elements of the statistical operator in the position representation are heavily suppressed. It is in particular seen as the recoilless limit of the so-called Caldeira Leggett ME

$$\mathrm{CL}\,[\hat{\rho}] = -\gamma \frac{2M}{\beta\hbar^2} \sum_{i=1}^{3} [\hat{\mathbf{x}}_i, [\hat{\mathbf{x}}_i, \hat{\rho}]] - \frac{i}{\hbar}\gamma \sum_{i=1}^{3} [\hat{\mathbf{x}}_i, \{\hat{\mathbf{p}}_i, \hat{\rho}\}], \qquad (2)$$

where γ is the friction coefficient, β the inverse temperature and M the mass of the microsystem. Equation (2) is usually considered for the description of quantum Brownian motion in the high temperature limit, which is not necessarily always the case in experimental setups where one wants to investigate coherence properties of the system and their washing out due to controlled or uncontrolled coupling to the external environment. The high temperature limit is linked to the fact that (2), in contrast with (1), cannot be cast into Lindblad form and therefore does not preserve positivity of the time evolution. A further term of the form $-\chi\gamma\frac{\beta}{M}\sum_{i=1}^{3}[\hat{\mathbf{p}}_i, [\hat{\mathbf{p}}_i, \hat{\rho}]]$ with $\chi \geq \frac{1}{8}$ is necessary in order to preserve complete positivity (Isar *et al.*, 1994; Vacchini, 2001a), a term which due to its different β dependence can be neglected in the high temperature limit. Different values of the coefficient χ have appeared in different models (Diósi, 1993a,b, 1995), but it appears that the correct value should be the *minimal* correction $\chi = \frac{1}{8}$. The friction term in (2) accounts for energy transfer and therefore thermalization of the Brownian particle, leading to the existence of a stationary solution of the form $e^{-\beta\frac{\hat{\mathbf{p}}^2}{2M}}$; though the typical time scales for decoherence and relaxation in this kind of models may easily differ by orders of magnitude, so that thermalization takes place on a much longer time scale, it is nevertheless of interest to consider a possibly fully realistic description, where all physical processes can be correctly described. In fact as has been pointed out (Ballentine, 1991; Gallis, 1991) the ME (1) predicts a steady growth in energy for the microsystem. Shortly after the proposal (1) another ME

$$\mathrm{GRW}\,[\hat{\rho}] = -\lambda\left(\hat{\rho} - \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3\mathbf{s}\, e^{-\frac{\alpha}{2}(\hat{\mathbf{x}}-\mathbf{s})^2} \hat{\rho} e^{-\frac{\alpha}{2}(\hat{\mathbf{x}}-\mathbf{s})^2}\right) \qquad (3)$$

has been introduced by Ghirardi *et al.* (1986), which also predicts a steady grow in energy (though for the proposed values of the parameters α and λ the growth is actually by orders of magnitude insignificantly small). The result (3) stands however on a completely different footing, since it is not meant as an appropriate description of the dynamics of a microsystem interacting with some environment, but as a fundamental modification of Schrödinger's equation allowing to solve the measurement problem and can be obtained from the latter by the insertion of a stochastic correction corresponding to white noise (for a recent review see Bassi and Ghirardi, 2003).

The result of Joos and Zeh was later improved by Gallis and Fleming (1990), always neglecting recoil effects. The motivation for this further work was the observation that due to (1) the incoherent part of the time evolution induces a suppression of the off-diagonal matrix elements according to $\frac{\partial}{\partial t}\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle = -\Lambda|\mathbf{x} - \mathbf{y}|^2\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle$, where the localization factor grows without bound for $|\mathbf{x} - \mathbf{y}|$ going to infinity. On physical grounds it is expected that such a behavior might hold at short length-scale, i.e., small $|\mathbf{x} - \mathbf{y}|$, while for long length-scale there should be no dependence on the spatial separation, otherwise the environment would have to be self-correlated over an infinite length scale. This unphysical feature does not appear in the model of quantum mechanics with spontaneous localization, in fact according to (3) one would have

$$\frac{\partial}{\partial t}\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle = -\lambda\frac{\alpha}{4}|\mathbf{x} - \mathbf{y}|^2\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle \quad \text{and} \quad \frac{\partial}{\partial t}\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle = -\lambda\langle \mathbf{x}|\hat{\rho}|\mathbf{y}\rangle \tag{4}$$

for short and long length-scales respectively, so that the localization effect saturates. It is to be pointed out that the quantity which actually distinguishes the two regimes is of the form $\mathbf{q}\cdot\hat{\mathbf{x}}$, where $\mathbf{q}$ is a typical value of momentum transfer corresponding to the relevant scattering dynamics, thus depending on details of microsystem, environment and their interaction potential, while $\hat{\mathbf{x}}$ are the position operators for the microsystem, thus depending on the considered matrix element. The result obtained by Gallis and Fleming is

$$\mathrm{GF}\,[\hat{\rho}] = \int d^3\mathbf{q}d^3\mathbf{q}'\,\frac{g(q)}{2q^4}\delta(q - q')|f(\mathbf{q},\mathbf{q}')|^2\left(e^{\frac{i}{\hbar}(\mathbf{q}-\mathbf{q}')\cdot\hat{\mathbf{x}}}\hat{\rho}\,e^{-\frac{i}{\hbar}(\mathbf{q}-\mathbf{q}')\cdot\hat{\mathbf{x}}} - \hat{\rho}\right), \tag{5}$$

with $g(q) = n(q)v(q)$, where $n(q)$ is the number density of scattering particles with momentum q, $v(q)$ their speed and $f(\mathbf{q},\mathbf{q}')$ the scattering amplitude. Considering the expression of GF one can check that it actually leads to results analogous to (4). Indeed the results (5) and (1) are considered as a standard reference for the study of decoherence, and they have been recently exploited (Alicki, 2002b; Viale *et al.*, 2003) in trying to quantitatively estimate decoherence in interference experiments with fullerene molecules. In (Alicki, 2002b) the connection between the models in (Joos and Zeh, 1985; Gallis and Fleming, 1990) and the theory of dynamical semigroups is considered, and the ME

$$A[\hat{\rho}] = \int d^3\mathbf{k}\,\tau(\mathbf{k})\left(e^{\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}}\hat{\rho}\,e^{-\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}} - \hat{\rho}\right) \tag{6}$$

is proposed, where $\tau(\mathbf{k})$ is the density of collisions per unit time leading to a momentum transfer $\mathbf{k}$. The operator structure and the role of the momentum transfer in its determination is here put in major evidence. This result can be easily connected to (5) by observing that setting $|f(\mathbf{q},\mathbf{q}')| = |f(\mathbf{q}-\mathbf{q}')| \equiv |f(\mathbf{k})|$ one has $\tau(\mathbf{k}) \equiv |f(\mathbf{k})|^2 \int \frac{d^3\mathbf{q}}{2q^4}\,g(q)\delta(q - |\mathbf{q}-\mathbf{k}|)$. In both (5) and (6) only the position operators $\hat{\mathbf{x}}$ appear, showing up in the typical expression $e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$, this unitary operator being strictly related to TI, as we shall see in Section 3.

The absence of the momentum operator in (5) and (6) indicates that neither can describe the approach to thermal equilibrium, and in fact similar to (1) they both predict a steady growth in energy for the microsystem, neglecting the effect of recoil in collisions. If only small momentum transfers are of relevance, or one assumes $\hat{\rho}$ diagonal enough in position representation, the unitary operators $e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$ can be expanded up to second order, leading from (5) or (6) to (1), where typical structures of double commutators with the position operators appear, corresponding to a Gaussian, diffusive behavior. ME like (1) or (2) can all be obtained starting from the general Lindblad structure $L[\hat{\rho}] = \sum_i [\hat{V}_i \hat{\rho} \hat{V}_i^\dagger - \frac{1}{2}\{\hat{V}_i^\dagger \hat{V}_i, \hat{\rho}\}]$ and making the Ansatz: $\hat{V}_i = \alpha_i \hat{\mathbf{p}} + \beta_i \hat{\mathbf{x}}$ (Isar *et al.*, 1994).

To cope with friction and thermalization to a suitable stationary state one has to modify (5) or (6) in order to let the momentum operators of the microsystem $\hat{\mathbf{p}}$ appear, similar to the modification in going from (1) to (2). The correction must be such that one has a suitable thermal stationary state and that energy of the microsystem does not grow to infinity. A first significant step in this direction has been done by Gallis (1993) with a phenomenological approach which always takes as starting point the formal Lindblad structure, but rather than the previous Ansatz assumes the more general expression $\hat{V}(\mathbf{q}) = \alpha(q) e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} + \beta(q) e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \mathbf{q}\cdot\hat{\mathbf{p}}$, substituting the sum over i with an integral over the momentum $\mathbf{q}$, already putting into evidence the unitary operator $e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$ which played such an important role in (5) and (6). The result is

$$G[\hat{\rho}] = \int d^3\mathbf{q}\, |\alpha(q)|^2 \big(e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \hat{\rho} e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \hat{\rho}\big)$$
$$+ \int d^3\mathbf{q}\, |\beta(q)|^2 \left(e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \mathbf{q}\cdot\hat{\mathbf{p}}\, \hat{\rho}\, \mathbf{q}\cdot\hat{\mathbf{p}}\, e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \frac{1}{2}\{(\mathbf{q}\cdot\hat{\mathbf{p}})^2, \hat{\rho}\} \right)$$
$$- \int d^3\mathbf{q}\, e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \big(\Re[\alpha^*(q)\beta(q)]\{\mathbf{q}\cdot\hat{\mathbf{p}}, \hat{\rho}\} + \Im[\alpha^*(q)\beta(q)][\mathbf{q}\cdot\hat{\mathbf{p}}, \hat{\rho}]\big)\, e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}},$$

and under certain restrictions on the phenomenological functions $\alpha(q)$ and $\beta(q)$ does in fact predict relaxation to thermal equilibrium. Further work in this direction has been done by Diósi (1995) starting from an analogy with the classical linear Boltzmann equation. He tried to connect similar structures of ME, in which both position and momentum operator of the microsystem appear, to an underlying dynamics in terms of collisions obtaining the result

$$D[\hat{\rho}] = \frac{nm^3}{\mu^5} \int d^3\mathbf{q} d^3\,\mathbf{q}'\, \delta(E(q) - E(q')) |f(\mathbf{q}, \mathbf{q}')|^2 \left(\hat{V} \hat{\rho} \hat{V}^\dagger - \frac{1}{2}\{\hat{V}^\dagger \hat{V}, \hat{\rho}\} \right) \tag{7}$$

with $\hat{V} = \sqrt{\sigma(\mathbf{q} + \frac{m}{M}(\hat{\mathbf{p}} + \mathbf{q}))} e^{\frac{i}{\hbar}(\mathbf{q}-\mathbf{q}')\cdot\hat{\mathbf{x}}}$, M mass of the microsystem, m the mass of the gas particles, μ the reduced mass, n the gas density, $E(q) = \frac{q^2}{2M}$ and σ the momentum distribution of the gas particles. If σ is given by a Boltzmann distribution an operator of the form $e^{-\beta\frac{\hat{\mathbf{p}}^2}{2M}}$ is a stationary solution of (7). A general result for a ME describing the motion of a particle interacting through collisions with some surrounding environment has been recently obtained starting from a scattering theory derivation (Vacchini, 2000, 2001a,b, 2002a; Lanz and Vacchini, 2002). The result relies on the appearance of a two-point correlation function known as dynamic structure factor, operator valued due to its dependence on the momentum operators of the microsystem. The dynamic structure factor obeys the detailed balance condition and therefore grants the existence of the expected stationary solution on very general grounds. The ME is

$$V[\hat{\rho}] = (2\pi)^4 \hbar^2 n \int d^3\,\mathbf{q}\, |\tilde{t}(q)|^2 \Bigg[e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \sqrt{S(\mathbf{q}, \hat{\mathbf{p}})}\, \hat{\rho} \sqrt{S(\mathbf{q}, \hat{\mathbf{p}})}\, e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \frac{1}{2}\left\{S(\mathbf{q}, \hat{\mathbf{p}}), \hat{\rho}\right\} \Bigg],$$

with $\tilde{t}(q)$ Fourier transform of the T-matrix describing the microphysical collisions and $S(\mathbf{q}, \mathbf{p})$ the positive two-point correlation function

$$S(\mathbf{q}, E) = \frac{1}{2\pi\hbar} \int dt \int d^3\mathbf{x}\, e^{\frac{i}{\hbar}[E(\mathbf{q},\mathbf{p})t - \mathbf{q}\cdot\mathbf{x}]} \frac{1}{N} \int d^3\mathbf{y}\, \langle N(\mathbf{y}) N(\mathbf{y}+\mathbf{x}, t)\rangle$$

with $S(\mathbf{q}, \mathbf{p}) \equiv S(\mathbf{q}, E)$, $E(\mathbf{q}, \mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2}{2M} - \frac{\mathbf{p}^2}{2M}$, $\mathbf{q}$ and E being momentum and energy transfer, while $N(\mathbf{y})$ is the particle density operator in the environment.

3. COVARIANCE PROPERTIES

The validity of a ME for the description of the reduced dynamics of a microsystem interacting with some environment ultimately rests on how realistic the environment and its coupling to the quantum system of interest have been described and how severe the approximations allowing for the derivation of the ME for the reduced system actually are. It is nevertheless of interest, and of guidance in determining equations giving the time evolution of the statistical operator, to check whether some general features, which should be common to any dynamical evolution, are actually present. Among these features one has preservation of trace and positivity of the statistical operator; complete positivity which emerges as a typical feature of quantum mechanics related to the non commutativity of the algebra of observables (Holevo, 2001); preservation of typical symmetries of the environment such as homogeneity (Kohen *et al.*, 1997) and in general invariance under the action of a group expressing some symmetry of the whole physical system; existence and uniqueness of a suitable stationary state with a canonical structure; correct description of the time evolution of the observables relevant to the dynamics, such as energy.

The most widespread approach is to start from or compare with the Lindblad structure of a ME, both in presence of bounded and unbounded operators, so that complete positivity and therefore in particular positivity is granted, and the same goes for preservation of the trace. In the case in which the physical system is characterized by some non trivial symmetry group however, one can rely on more recent and refined results than the one by Lindblad. The possible structures of generators of quantum dynamical semigroups covariant under the action of a symmetry group have been characterized in particular in the case of the two Abelian Lie groups R and $U(1)$ (Holevo, 1995, 1996), also taking care of defining a suitable domain in the case in which the relevant operators are unbounded.

Since we are focusing on structures of ME describing the loss of coherence of a microsystem interacting through collisions with a homogeneous environment, we will consider in some detail only the structure of the generator of a TI quantum dynamical semigroup. This result has been settled by Holevo and gives a non-commutative quantum generalization of the Lévy–Khintchine formula. In the following we will try to briefly summarize Holevo's results.

Let us first consider the case of a norm-continuous conservative quantum dynamical semigroup $\{\Phi_t; t \geq 0\}$ acting on the algebra of bounded operators in $L^2(\mathbf{R}^3)$, whose generator $\mathcal{L}$ is a completely dissipative map satisfying

$$\frac{d}{dt}\Phi_t[\hat{X}] = \mathcal{L}[\Phi_t[\hat{X}]] \quad X \in \mathcal{B}(L^2(\mathbf{R}^3)) \quad t \geq 0 \tag{8}$$

with $\Phi_0[\hat{X}] = \hat{X}$ and $\Phi_t[\hat{I}] = \hat{I}$ due to conservativity, i.e., trace preservation. If Φ_t is norm-continuous the generator $\mathcal{L}$ admits a *standard representation*

$$\mathcal{L}[\hat{X}] = \frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \Psi[\hat{X}] - \frac{1}{2}\{\Psi[\hat{I}], \hat{X}\} \quad \hat{X} \in \mathcal{B}(L^2(\mathbf{R}^3)) \tag{9}$$

with Ψ a normal completely positive map and $\hat{H}$ self-adjoint. The semigroup is said to be covariant under the action of a unitary representation $\hat{U}(\mathbf{a}) = e^{\frac{i}{\hbar}\mathbf{a}\cdot\hat{\mathbf{x}}\hat{\mathbf{p}}}$, $\mathbf{a} \in \mathbf{R}^3$ of the group of translations, i.e., translation-covariant, provided

$$\Phi_t[\hat{U}^\dagger(\mathbf{a})\hat{X}\hat{U}(\mathbf{a})] = \hat{U}^\dagger(\mathbf{a})\Phi_t[\hat{X}]\hat{U}(\mathbf{a}) \quad \hat{X} \in \mathcal{B}(L^2(\mathbf{R}^3)) \quad \mathbf{a} \in \mathbf{R}^3 \quad t \geq 0 \tag{10}$$

holds. If $\mathcal{L}$ is covariant in the sense of (10) in the decomposition (9) the map Ψ can always be chosen covariant and $\hat{H}$ commuting with the unitary representation $\hat{U}(\mathbf{q})$. Under the restriction (10) the general structure of the bounded generator of the semigroup in (8) is in fact given by

$$\begin{aligned}\mathcal{L}[\hat{X}] = {} & \frac{i}{\hbar}[H(\hat{\mathbf{p}}), \hat{X}] \\ & + \int \sum_{j=1}^{\infty} \Big[L_j^\dagger(\mathbf{q}, \hat{\mathbf{p}})\hat{U}^\dagger(\mathbf{q})\hat{X}\hat{U}(\mathbf{q})L_j(\mathbf{q}, \hat{\mathbf{p}}) \\ & - \frac{1}{2}\{L_j^\dagger(\mathbf{q}, \hat{\mathbf{p}})L_j(\mathbf{q}, \hat{\mathbf{p}}), \hat{X}\} \Big] d\mu(\mathbf{q})\end{aligned} \tag{11}$$

where $\hat{U}(\mathbf{q}) = e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$, $H(\cdot) = H^*(\cdot)$, $L_j(\mathbf{q}, \cdot)$ are bounded functions, $\mu(\mathbf{q})$ is a positive σ-finite measure on $\mathbf{R}^3$ and $\int \sum_{j=1}^{\infty} |L_j(\mathbf{q}, \cdot)|^2 d\mu(\mathbf{q}) < +\infty$.

In the case in which the family of maps $\{\Phi_t; t \geq 0\}$ acting on $\mathcal{B}(L^2(\mathbf{R}^3))$ are generally unbounded it is convenient to consider the equation

$$\frac{d}{dt}\langle\phi|\Phi_t[\hat{X}]\psi\rangle = \mathcal{L}(\phi; \Phi_t[\hat{X}]; \psi) \quad \hat{X} \in \mathcal{B}(L^2(\mathbf{R}^3)) \quad t \geq 0$$

with $\Phi_0[\hat{X}] = \hat{X}$, $\Phi_t[\hat{I}] = \hat{I}$ and $\phi, \psi \in \mathcal{D} \subset L^2(\mathbf{R}^3)$, where $\mathcal{D}$ is some dense domain. The expression $\mathcal{L}(\phi; \hat{X}; \psi)$ is the so-called *form-generator*, i.e., a function of $\phi, \psi \in \mathcal{D} \subset L^2(\mathbf{R}^3)$ and $\hat{X} \in \mathcal{B}(L^2(\mathbf{R}^3))$ characterized by the following basic properties: (1) $\mathcal{L}(\phi; \hat{X}; \psi)$ is linear in $\hat{X}$ and ψ, anti-linear in ϕ and such that $\mathcal{L}^*(\phi; \hat{X}; \psi) = \mathcal{L}(\psi; \hat{X}^\dagger; \phi)$; (2) for all finite subsets $\{\psi_j\} \in \mathcal{D} \subset L^2(\mathbf{R}^3)$ and $\{\hat{X}_j\} \in \mathcal{B}(L^2(\mathbf{R}^3))$ such that $\sum_j \hat{X}_j\psi_j = 0$ one has $\sum_{jk} \mathcal{L}(\psi_j; \hat{X}_j^\dagger \hat{X}_k; \psi_k) \geq 0$ (conditional complete positivity); (3) $\mathcal{L}(\phi; \hat{I}; \psi) = 0 \; \forall \phi, \psi \in \mathcal{D} \subset L^2(\mathbf{R}^3)$ (conservativity); together with suitable continuity properties. The form-generator also admits a *standard representation*

$$\begin{aligned}\mathcal{L}(\phi; \hat{X}; \psi) = &\sum_j \langle \hat{L}_j\phi|\hat{X}\hat{L}_j\psi\rangle \\ &- \langle\hat{K}\phi|\hat{X}\psi\rangle - \langle\phi|\hat{X}\hat{K}\psi\rangle \quad \phi, \psi \in \mathcal{D}, \; \hat{X} \in \mathcal{B}(L^2(\mathbf{R}^3))\end{aligned}$$

with $\hat{K}$ and $\hat{L}_j$ densely defined operators. The covariance condition analogous to (10) is now expressed by $\mathcal{L}(\phi; \hat{U}^\dagger(\hat{\mathbf{a}})\hat{X}\hat{U}(\hat{\mathbf{a}}); \psi) = \mathcal{L}(\hat{U}(\hat{\mathbf{a}})\phi; \hat{X}; \hat{U}^\dagger(\hat{\mathbf{a}})\psi)$, together with the invariance of the domain $\mathcal{D}$ under the unitary representation. It is however no longer possible to put into evidence an Hamiltonian contribution commuting with the unitary representation and a completely positive map. Taking as domain the space of twice continuously differentiable functions with compact support in the momentum representation of the CCR, i.e., $\mathcal{D} = C_0^2(\mathbf{R}^3)$, and asking for suitable continuity properties the general structure of the TI form-generator is given by $\mathcal{L} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{P}}$, where $\mathcal{L}_{\mathrm{G}}$ is the Gaussian, continuous component corresponding to the formal operator expression

$$\mathcal{L}_{\mathrm{G}}[\hat{X}] = \frac{i}{\hbar}[\hat{\mathbf{y}}_0 + H(\hat{\mathbf{p}}), \hat{X}] + \sum_{k=1}^{3} (\hat{\mathbf{y}}_k + L_k(\hat{\mathbf{p}}))^\dagger \, \hat{X} \, (\hat{\mathbf{y}}_k + L_k(\hat{\mathbf{p}})) - \hat{K}^\dagger\hat{X} - \hat{X}\hat{K}$$

$$K = \frac{1}{2}\sum_{k=1}^{r} \left(\hat{\mathbf{y}}_k^2 + 2\hat{\mathbf{y}}_k L_k(\hat{\mathbf{p}}) + L_k^\dagger(\hat{\mathbf{p}})L_k(\hat{\mathbf{p}})\right) \tag{12}$$

with $\hat{\mathbf{y}}_k = \sum_{i=1}^{3} a_{ki}\hat{\mathbf{x}}_i$, $k = 0, \dots, 3$, $a_{ki} \in \mathbf{R}$, $H(\cdot) = H^*(\cdot) \in L^2_{\mathrm{loc}}(\mathbf{R}^3)$ and $|L_k(\cdot)|^2 \in L^2_{\mathrm{loc}}(\mathbf{R}^3)$, while $\mathcal{L}_{\mathrm{P}}$ is the Poisson, jump component

$$\begin{aligned}
\mathcal{L}_{\mathrm{P}}[\hat{X}] = \int \sum_{j=1}^{\infty} \Bigg[& L_j^\dagger(\mathbf{q}, \hat{\mathbf{p}}) \hat{U}^\dagger(\mathbf{q}) \hat{X} \hat{U}(\mathbf{q}) L_j(\mathbf{q}, \hat{\mathbf{p}}) \\
& - \frac{1}{2}\{L_j^\dagger(\mathbf{q}, \hat{\mathbf{p}}) L_j(\mathbf{q}, \hat{\mathbf{p}}), \hat{X}\} \Bigg] d\mu(\mathbf{q}) \\
& + \int \sum_{j=1}^{\infty} [\omega_j(\mathbf{q}) L_j^\dagger(\mathbf{q}, \hat{\mathbf{p}}) (\hat{U}^\dagger(\mathbf{q}) \hat{X} \hat{U}(\mathbf{q}) - \hat{X}) \\
& + (\hat{U}^\dagger(\mathbf{q}) \hat{X} \hat{U}(\mathbf{q}) - \hat{X}) L_j(\mathbf{q}, \hat{\mathbf{p}}) \omega_j^*(\mathbf{q})] d\mu(\mathbf{q}) \\
& + \int \sum_{j=1}^{\infty} \left[\hat{U}^\dagger(\mathbf{q}) \hat{X} \hat{U}(\mathbf{q}) - \hat{X} - i \frac{[\hat{X}, \hat{\mathbf{x}y} \cdot \mathbf{q}]}{1 + |\mathbf{q}|^2} \right] |\omega_j(\mathbf{q})|^2 d\mu(\mathbf{q}) \quad (13)
\end{aligned}$$

with $\mu(\mathbf{q})$ a positive σ-finite measure on $\mathbf{R}^3$, $\omega_j(\mathbf{q})$ complex measurable functions and the further conditions $\int |\mathbf{q}|^2/(1 + |\mathbf{q}|^2) \sum_{j=1}^{\infty} |\omega_j(\mathbf{q})|^2 d\mu(\mathbf{q}) < +\infty$ and $\int \sum_{j=1}^{\infty} |L_j(\mathbf{q}, \cdot)|^2 d\mu(\mathbf{q}) \in L^2_{\mathrm{loc}}(\mathbf{R}^3)$, Eqs. (12) and (13) giving a non-commutative quantum generalization of the Lévy–Khintchine formula. Despite appearance the result can still be cast in Lindblad form.

All the ME for the statistical operator considered in the previous section can be formally compared to the pre-adjoint of the maps given in (11) or (12) and (13), with suitable choices of parameters and functions (Vacchini, 2001a, 2002b). One thus sees that not all the ME proposed in Section 2. are proper generators of quantum dynamical semigroups, as already mentioned in connection with the property of complete positivity. In particular one can now clearly distinguish between Gaussian and Poisson components. Equations (12) and (13) also give some hints about the possible structures of the ME for a microsystem interacting with a homogeneous environment which might be derived in future research work, starting form detailed physical models. Of course the case of a microsystem interacting through collisions with a TI bath is just one of the possible physical models interesting for the study of decoherence, another most important example is the interaction of a charged particle with the electromagnetic field and the related phenomenon of decoherence due to Bremsstrahlung (Breuer and Petruccione, 2001).

Note added. After completion of the first version of the manuscript, further work deserving attention has been done on the subject (Dodd and Halliwell, 2003; Hornberger and Sipe, 2003), though from a different standpoint.

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